ELSEVIER

Available online at www.sciencedirect.com
ScienceDirect

Journal of Approximation Theory 160 (2009) 243–255

Journal of Approximation Theory
www.elsevier.com/locate/jat

Convergence of iterates of genuine and ultraspherical Durrmeyer operators to the limiting semigroup: C^2-estimates

Heiner Gonska[a], Margareta Heilmann[b,*], Ioan Raşa[c]

[a] *Department of Mathematics, University of Duisburg-Essen, Forsthausweg 2, D-47057 Duisburg, Germany*
[b] *Faculty of Mathematics and Natural Sciences, University of Wuppertal, Gauß straß e 20, D-42119 Wuppertal, Germany*
[c] *Department of Mathematics, Technical University, Str. C. Daicoviciu, 15, RO-400020 Cluj-Napoca, Romania*

Received 19 September 2007; received in revised form 2 November 2008; accepted 3 November 2008

Communicated by Yuan Xu
Available online 12 November 2008

Dedicated to Professor Paul L. Butzer on the occasion of his 80th birthday

Abstract

In the present note a general inequality for the degree of approximation of semigroups by iterates of commuting bounded linear operators on Banach spaces is given. Combining this with a recent quantitative Voronovskaja-type result applications to Durrmeyer operators with ultraspherical weights are derived. Our considerations include the genuine Bernstein–Durrmeyer operators.

MSC: 20Mxx; 41A10; 41A17; 41A25; 41A36

Keywords: Operators on Banach spaces; Semigroup; Bernstein–Durrmeyer operators with ultraspherical weights; Degree of approximation; Genuine Bernstein–Durrmeyer operators; Moduli of smoothness

1. Introduction

In 1970 Karlin and Ziegler [21] proved that iterated Bernstein operators $B_n^{k_n}$ converge to the elements $T(t)$, $0 \leqslant t \leqslant \infty$, of a positive contraction semigroup on $C[0, 1]$ with infinitesimal

* Corresponding author.
E-mail addresses: heiner.gonska@uni-due.de (H. Gonska), heilmann@math.uni-wuppertal.de (M. Heilmann), Ioan.Rasa@math.utcluj.ro (I. Raşa).

0021-9045/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2008.11.002

generator

$$A(f;x)=\begin{cases}\dfrac{x(1-x)}{2}f''(x) & \text{if } 0<x<1,\\ 0 & \text{if } x=0,1.\end{cases}$$

The sufficient condition is that $\lim_{n\to\infty}\frac{k_n}{n}\to t$.

The assertion in the article mentioned is non-quantitative. The first full quantitative version of it was given by Gonska and Raşa in [19] for the classical Bernstein and by Kacsó [20] for the genuine Bernstein–Durrmeyer operators, and—with a different proof—also in [17]. All estimates mentioned are based on estimates for functions in $C^4[0,1]$. In [19] the authors conjectured that it is possible to extend the problem to the consideration of elements of $C^2[0,1]$. A first step into this direction was done by Mangino and Raşa [23] who showed that $C^3[0,1]$ can be used. Moreover, one recent article and one preprint of Campiti and Tacelli [11,10] show that it is possible to use $C^{2,\alpha}[0,1]=\{f\in C^2[0,1]: f''\in Lip_\alpha\}$, $0<\alpha<1$, for the basic estimates.

The results in [19,23] were inspired by Butzer and his school (see, e.g., [7–13]). As far as the theory of semigroups is concerned, useful references are [14,29].

In this note we continue our research by combining the technique of [23] with the quantitative Voronovskaja result from [18]. This will show that for commuting operators it is indeed sufficient to consider $C^2[0,1]$ to give the basic inequality. The Voronovskaja statement is an important tool to obtain our propositions below. It should therefore be mentioned that two recent articles by Gonska [16] and Tachev [28] continue the research in [18].

The organization of this paper is as follows: Section 2 contains a general estimate for the degree of approximation of semigroup elements by iterates of mutually commuting bounded linear operators. Section 3 recalls the quantitative Voronovskaja-type result needed in the applications. General applications for ultraspherical and genuine Durrmeyer operators are given in Section 4, but only for sufficiently large values of the operator index n. Part of the computations and arrangements were done with the help of MAPLE and only later checked by hand. For cases of particular interest, namely those of the genuine Bernstein–Durrmeyer operator ($\alpha=-1$) and for the Chebyshev case of the first kind ($\alpha=-\frac{1}{2}$), the Legendre case ($\alpha=0$) and for the Chebyshev case of the second kind ($\alpha=\frac{1}{2}$) our results for big values of n are supplemented for small ones in order to complete the picture. Section 6 shows how to carry the estimates for C^2-functions over to functions in $C[0,1]$. It shows also that—as far as order of approximation is concerned—nothing gets lost.

2. A general estimate for commuting operators

Let $(X,\|\cdot\|)$ be a Banach space and (L_n), $n\in\mathbb{N}$, a sequence of bounded linear operators on X such that for each $n,m\in\mathbb{N}$

$$\|L_n\|=1 \quad\text{and}\quad L_mL_n=L_nL_m. \tag{1}$$

We shall suppose that:

(i) There exists $Ax:=\lim_{n\to\infty}n(L_nx-x)$ for x in a dense linear subspace $D(A)$.

(ii) $(A,D(A))$ is closable and the closure $(\overline{A},D(\overline{A}))$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X.

(iii) For each $t \geqslant 0$ and each sequence $(k_n)_{n \geqslant 1}$ of positive integers such that $\lim_{n\to\infty} \frac{k_n}{n} = t$ we have $\lim_{n\to\infty} L_n^{k_n} x = T(t)x$, $x \in X$.

The approximation of semigroups, as in (iii), by iterates of *positive* (not necessarily commuting) linear operators was intensively investigated by Francesco Altomare and his school; see [2–4] and the references therein. The significance of (i)–(iii) (in particular the role of the closability) is illustrated by Theorems 6.2.6 and 6.3.5 in [2]. As consequences of (1) and (iii) we get

$$\|T(t)\| \leqslant 1, \quad t \geqslant 0, \tag{2}$$

$$L_n T(t) = T(t) L_n, \quad n \geqslant 1, \ t \geqslant 0. \tag{3}$$

Here are some examples of sequences of operators L_n satisfying the above assumptions:

(a) Durrmeyer operators with Jacobi weights on [0, 1],
(b) Durrmeyer operators with ultraspherical weights on [0, 1],
(c) Bernstein–Durrmeyer operators on [0, 1],
(d) Genuine Bernstein–Durrmeyer operators on [0, 1].

The basic estimate for all further applications is the following general theorem concerning the difference between powers of L_n and the members of the corresponding semigroup. The proof is inspired by that of Theorem 3 in the paper [23] by Mangino and Raşa, and the new ingredient is the commutativity.

Theorem 1. *Let $n, k \in \mathbb{N}$, $c \geqslant 0$ a real constant, $t \geqslant 0$ real and $x \in D(\overline{A})$. Then*

$$\|L_n^k x - T(t)x\| \leqslant t\|(A_n - \overline{A})x\| + \left(\left| t - \frac{k}{n+c} \right| + \frac{\sqrt{k}}{n+c} \right) \|A_n x\|, \tag{4}$$

where $A_n := (n + c)(L_n - I)$.

Proof. We first claim that

$$\|e^{tA_n}\| \leqslant 1, \quad t \geqslant 0. \tag{5}$$

Indeed,

$$e^{tA_n} = e^{-(n+c)t} \sum_{i=0}^{\infty} \frac{[(n+c)t]^i}{i!} L_n^i \quad \text{and hence } \|e^{tA_n}\| \leqslant e^{-(n+c)t} \sum_{i=0}^{\infty} \frac{[(n+c)t]^i}{i!} = 1.$$

On the other hand,

$$\begin{aligned} e^{tA_n} x - T(t)x &= -\int_0^t \frac{d}{ds} [e^{(t-s)A_n} T(s)x]\, ds \\ &= \int_0^t [A_n e^{(t-s)A_n} T(s)x - e^{(t-s)A_n} \overline{A} T(s)x]\, ds \\ &= \int_0^t e^{(t-s)A_n} (A_n - \overline{A}) T(s)x\, ds. \end{aligned}$$

Since $x \in D(\overline{A})$, we have $\overline{A} T(s)x = T(s)\overline{A}x$. Moreover, according to (3), $A_n T(s)x = T(s) A_n x$. Thus $e^{tA_n} x - T(t)x = \int_0^t e^{(t-s)A_n} T(s)(A_n - \overline{A})x\, ds$.

Taking into account (2) and (5) we get

$$\|e^{tA_n}x - T(t)x\| \leqslant t\|(A_n - \overline{A})x\|. \tag{6}$$

We also have

$$\|L_n^k x - T(t)x\| \leqslant \|T(t)x - e^{tA_n}x\| + \|e^{tA_n}x - e^{k/(n+c)A_n}x\| + \|e^{k/(n+c)A_n}x - L_n^k x\|.$$

From the book by Engel and Nagel [14, Chapter III, Lemma 5.1] we infer that

$$\|e^{k/(n+c)A_n}x - L_n^k x\| \leqslant \frac{\sqrt{k}}{n+c}\|A_n x\|. \tag{7}$$

Moreover,

$$\|e^{tA_n}x - e^{k/(n+c)A_n}x\| = \left\|\int_{k/(n+c)}^{t} e^{sA_n}A_n x\,ds\right\| \leqslant \left|t - \frac{k}{n+c}\right|\|A_n x\|. \tag{8}$$

Combining (6)–(8) we obtain (4). □

The last inequalities in the proof of Theorem 1 show that, in order to come up with the final estimate, we need two things:

(i) Quantitative Voronovskaja-type inequalities to estimate $\|(A_n - \overline{A})(x)\|$.
(ii) Quantitative assertions concerning $\|A_n x\|$, but only for such x for which the Voronovskaja inequality holds. Normally these are well-known inequalities.

3. A quantitative Voronovskaja-type result

We write $e_i(t) = t^i$, $i \in \mathbb{N}_0$, for the i-th monomial.

One important ingredient for the inequalities below is the following quantitative Voronovskaja-type theorem (see [16, Theorem 3.2]).

Theorem 2. *Let $L : C[0,1] \longrightarrow C[0,1]$ be a positive linear operator satisfying $Le_0 = e_0$, and $f \in C^2[0,1]$. Then*

$$\begin{aligned}
&|Lf(x) - f(x) - L(e_1 - xe_0)(x)f'(x) - \frac{1}{2}L(e_1 - xe_0)^2(x)f''(x)| \\
&\quad \leqslant \frac{1}{2}L(e_1 - xe_0)^2(x)\tilde{\omega}\left(f''; \frac{1}{3}\cdot\frac{L(|e_1 - xe_0|^3)(x)}{L(e_1 - xe_0)^2(x)}\right) \\
&\quad \leqslant \frac{1}{2}L(e_1 - xe_0)^2(x)\tilde{\omega}\left(f''; \frac{1}{3}\cdot\sqrt{\frac{L(e_1 - xe_0)^4(x)}{L(e_1 - xe_0)^2(x)}}\right).
\end{aligned}$$

Here $\tilde{\omega}(g;\varepsilon)$ is the least concave majorant of the first order modulus of continuity given by

$$\tilde{\omega}(g;\varepsilon) = \begin{cases} \sup\limits_{0\leqslant x\leqslant\varepsilon\leqslant y\leqslant 1} \dfrac{(\varepsilon - x)\omega(g;y) + (y-\varepsilon)\omega(g;x)}{y-x}, & 0\leqslant\varepsilon\leqslant 1, \\ \omega(g;1), & \varepsilon > 1. \end{cases}$$

Remark 3. We remark the following.

(i) The quantity $L(|e_1 - xe_0|^3)(x)$ is normally very inconvenient to be computed. Therefore we used the Cauchy–Schwarz inequality to derive the last upper bound.
(ii) One property of $\tilde{\omega}$ following directly from the definition is that for $g \in C^1[0,1]$ one has

$$\tilde{\omega}(g;\varepsilon) \leqslant \varepsilon \|g'\|.$$

This inequality will be used below.

4. Applications to Durrmeyer operators with ultraspherical weights

Durrmeyer operators with Jacobi weights $w_{\alpha,\beta}$ are defined for $n \in \mathbb{N}$, $\alpha, \beta > -1$ real, by

$$M_{n,\alpha,\beta} f(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{1}{c_{n,k,\alpha,\beta}} \int_0^1 p_{n,k}(t) w_{\alpha,\beta}(t) f(t)\, dt,$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leqslant k \leqslant n,$$
$$c_{n,k,\alpha,\beta} := \int_0^1 p_{n,k}(t) w_{\alpha,\beta}(t)\, dt, \quad w_{\alpha,\beta}(t) := t^{\alpha}(1-t)^{\beta}.$$

They are defined for all functions f for which the integrals exist, see for example [24,5,6] for further details. For functions f with appropriate differentiability properties, asymptotic expansions for the polynomials $M_{n,\alpha,\beta} f$ and their derivatives as well are considered in [1]. The research on Durrmeyer operators was initiated by Derriennic [12] in 1981 for the case $\alpha = \beta = 0$. Here we are only interested in the ultraspherical cases $\alpha = \beta$ and the limiting case $\alpha \to -1$.

To be more specific, we consider the ultraspherical Durrmeyer operators $M_{n,\alpha}$ on $C[0,1]$ with the supremum norm. For $\alpha > -1$ these are defined by (see, e.g., [26])

$$M_{n,\alpha} f(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{1}{c_{n,k,\alpha}} \int_0^1 p_{n,k}(t) w_{\alpha}(t) f(t)\, dt,$$

where

$$c_{n,k,\alpha} := \int_0^1 p_{n,k}(t) w_{\alpha}(t)\, dt = \binom{n}{k} \frac{\Gamma(k+\alpha+1)\Gamma(n-k+\alpha+1)}{\Gamma(n+2\alpha+2)},$$
$$w_{\alpha}(t) := t^{\alpha}(1-t)^{\alpha},$$

and for $\alpha = -1$, by

$$M_{n,-1} f(x) = f(0) p_{n,0}(x) + f(1) p_{n,n}(x) + (n-1) \sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t) f(t)\, dt.$$

The latter are often denoted by U_n and called the genuine Bernstein–Durrmeyer operators. Note that for $n = 1$ the sum is empty, i.e., equal to 0. One very useful article about them is the one by Parvanov and Popov [25]; see also the dissertation by Sauer [27].

We will use the notations $X := x(1-x)$ and $X' := \frac{d}{dx} X = 1 - 2x$.

The associated semigroup $(T(t))_{t\geqslant 0}$ for $\alpha \geqslant -1$ is investigated (even in the case of Durrmeyer operators with Jacobi weights) in [4,26]. The asymptotic formula and the infinitesimal generator are (see also the work by Berens and Xu [4,26]):

$$\lim_{n\to\infty}(n+2\alpha+2)(M_{n,\alpha}f-f)=A_\alpha f,\quad f\in C^2[0,1],\tag{9}$$

where $A_\alpha f(x)=Xf''(x)+(\alpha+1)X'f'(x)$, $x\in[0,1]$. We stress the fact that this covers the case $\alpha=-1$.

We will use the following (Pochhammer) notation:

$$(y)_k=\prod_{l=0}^{k-1}(y+l)=\frac{\Gamma(y+k)}{\Gamma(y)}.$$

In the sequel we will use the following abbreviations.

$$\begin{aligned}
p_1 &:= p_1(\alpha,n) := n+\alpha+1-2(\alpha+1)_2,\\
p_2 &:= p_2(\alpha) := (\alpha+1)_2,\\
p_3 &:= p_3(\alpha,n) := 3n(\alpha+2)+3(\alpha+1)_2-2(\alpha+1)_3,\\
p_4 &:= p_4(\alpha,n) := 3n[n-4(\alpha+1)_2-6(\alpha+2)-1]+4(\alpha+1)_4\\
&\qquad\qquad -12(\alpha+1)_3+3(\alpha+1)_2,\\
p_5 &:= p_5(\alpha,n) := 3n(\alpha+2)_2-2(\alpha+1)_4+3(\alpha+1)_3,\\
p_6 &:= p_6(\alpha) := (\alpha+1)_4,\\
p_7 &:= p_7(\alpha) := (\alpha+1)_3.
\end{aligned}$$

With the recursion formula from [24] for the moments long and tedious computations confirm the following:

$$M_{n,\alpha}(e_1-xe_0)(x)=X'\frac{\alpha+1}{n+2\alpha+2},\tag{10}$$

$$M_{n,\alpha}(e_1-xe_0)^2(x)=\frac{2p_1X+p_2}{(n+2\alpha+2)_2},\tag{11}$$

$$M_{n,\alpha}(e_1-xe_0)^3(x)=X'\frac{2p_3X+p_7}{(n+2\alpha+2)_3},\tag{12}$$

$$M_{n,\alpha}(e_1-xe_0)^4(x)=\frac{4p_4X^2+4p_5X+p_6}{(n+2\alpha+2)_4}.\tag{13}$$

By using Taylor's formula, (10) and (11), we obtain for $p_1(\alpha,n)\geqslant 0$, i.e., $n\geqslant 2(\alpha+1)_2-(\alpha+1)$, the inequality

$$\|M_{n,\alpha}f-f\| \leqslant \frac{\alpha+1}{n+2\alpha+2}\|f'\|+\frac{1}{4(n+2\alpha+3)}\|f''\|\tag{14}$$

$$\leqslant \frac{1}{n+2\alpha+2}\left[(\alpha+1)\|f'\|+\frac{1}{4}\|f''\|\right],\quad f\in C^2[0,1].\tag{15}$$

Note that the condition $p_1(\alpha,n)\geqslant 0$ is needed for (14) and (15), since otherwise less favourable bounds are obtained.

In our next lemma we compute the norm of the ratio between the fourth and the second moments.

Lemma 4. *Let $\alpha \geqslant -1$, $x \in [0,1]$ and $n \geqslant N(\alpha) := 4(\alpha+1)_2 + 6(\alpha+2) + 1$.*
We have

$$\frac{M_{n,\alpha}(e_1 - xe_0)^4(x)}{M_{n,\alpha}(e_1 - xe_0)^2(x)} \leqslant \frac{M_{n,\alpha}(e_1 - xe_0)^4(\frac{1}{2})}{M_{n,\alpha}(e_1 - xe_0)^2(\frac{1}{2})} = \frac{3}{2}\,\frac{n+\alpha+2}{(n+2\alpha+4)_2}. \tag{16}$$

Proof. From (11) and (13) we get for the ratio between the fourth and the second moments

$$\frac{M_{n,\alpha}(e_1 - xe_0)^4(x)}{M_{n,\alpha}(e_1 - xe_0)^2(x)} = \frac{1}{(n+2\alpha+4)_2} \cdot \frac{4p_4X^2 + 4p_5X + p_6}{2p_1X + p_2}. \tag{17}$$

In order to estimate the ratio, we determine the extremal values over [0, 1]. For the first derivative we have

$$\frac{d}{dx}\left\{\frac{M_{n,\alpha}(e_1 - xe_0)^4(x)}{M_{n,\alpha}(e_1 - xe_0)^2(x)}\right\} = 2X'\frac{4p_1p_4X^2 + 4p_2p_4X + 2p_2p_5 - p_1p_6}{(n+2\alpha+4)_2[2p_1X + p_2]^2}. \tag{18}$$

Hence the first derivative has a zero at $x = \frac{1}{2}$.

We show next that the numerator on the right-hand side of (18) is positive for $n \geqslant N(\alpha)$ and $x \in [0,1]$. Since

$$4(\alpha+1)_4 - 12(\alpha+1)_3 + 3(\alpha+1)_2 = (\alpha+1)_2[3 + 4(\alpha+1)(\alpha+3)] \geqslant 0,$$

the assumption on n implies $p_4 \geqslant 0$ and also $p_1 \geqslant 0$. Because obviously $p_2 \geqslant 0$, it follows that

$$p_1p_4 \geqslant 0 \quad \text{and} \quad p_2p_4 \geqslant 0. \tag{19}$$

Furthermore, again using the assumption on n we obtain

$$\begin{aligned} 2p_2p_5 - p_1p_6 &= (\alpha+1)_3[n(5\alpha+8) - (\alpha+1)(2\alpha^2 + 7\alpha + 8)] \\ &> (\alpha+1)_3(\alpha+2)(9\alpha+16)(2\alpha+5) \geqslant 0. \end{aligned} \tag{20}$$

By (19) and (20) we get

$$4p_1p_4X^2 + 4p_2p_4X + 2p_2p_5 - p_1p_6 \geqslant 0$$

for each $\alpha \geqslant -1$, $x \in [0,1]$ and $n \geqslant 4(\alpha+1)_2 + 6(\alpha+2) + 1$.

Altogether we have shown that the ratio in question is monotonically increasing on $[0, \frac{1}{2}]$ and monotonically decreasing on $[\frac{1}{2}, 1]$ and that at $x = \frac{1}{2}$ there is a maximum. This proves our lemma by putting $x = \frac{1}{2}$ in (17). □

From Theorem 2 we derive by using (10), (11), (16)

$$\begin{aligned} &\left|(M_{n,\alpha}f - f)(x) - X'\frac{\alpha+1}{n+2\alpha+2}f'(x) - \frac{1}{2}\cdot\frac{2p_1X + p_2}{(n+2\alpha+2)_2}f''(x)\right| \\ &\quad\leqslant \frac{1}{2}M_{n,\alpha}(e_1 - xe_0)^2(x)\tilde{\omega}\left(f''; \frac{1}{3}\sqrt{\frac{M_{n,\alpha}(e_1 - xe_0)^4(x)}{M_{n,\alpha}(e_1 - xe_0)^2(x)}}\right) \\ &\quad\leqslant \frac{1}{2}\cdot\frac{2p_1X + p_2}{(n+2\alpha+2)_2}\tilde{\omega}\left(f''; \sqrt{\frac{1}{6}\cdot\frac{n+\alpha+2}{(n+2\alpha+4)_2}}\right). \end{aligned}$$

Moreover, one has

$$\begin{aligned}
&|(n+2\alpha+2)(M_{n,\alpha}f-f)(x)-A_\alpha f(x)|\\
&\quad=|(n+2\alpha+2)(M_{n,\alpha}f-f)(x)-(\alpha+1)X'f'(x)-Xf''(x)|\\
&\quad\leqslant\left|(n+2\alpha+2)(M_{n,\alpha}f-f)(x)-(\alpha+1)X'f'(x)-\frac{1}{2}\cdot\frac{2p_1X+p_2}{n+2\alpha+3}f''(x)\right|\\
&\qquad+\left|\frac{1}{2}\cdot\frac{(\alpha+1)_2-2(\alpha+2)(2\alpha+3)X}{n+2\alpha+3}f''(x)\right|.
\end{aligned}\tag{21}$$

As $n\geqslant N(\alpha)$ we have for all $x\in[0,1]:2p_1X+p_2\leqslant\frac{1}{2}(n+\alpha+1)$. Furthermore,

$$|\tfrac{1}{2}[(\alpha+1)_2-2(\alpha+2)(2\alpha+3)X]|\leqslant C(\alpha),$$

where

$$C(\alpha)=\begin{cases}\dfrac{(\alpha+1)_2}{2}, & \alpha>-\dfrac{1}{2},\\[2mm] \dfrac{\alpha+2}{4}, & -1\leqslant\alpha\leqslant-\dfrac{1}{2}.\end{cases}\tag{22}$$

So together with (21) we arrive at

$$\begin{aligned}
&\|(n+2\alpha+2)(M_{n,\alpha}f-f)-A_\alpha f\|\\
&\quad\leqslant\frac{1}{4}\cdot\frac{n+\alpha+1}{n+2\alpha+3}\tilde{\omega}\left(f'';\sqrt{\frac{1}{6}\cdot\frac{n+\alpha+2}{(n+2\alpha+4)_2}}\right)+\frac{C(\alpha)}{n+2\alpha+3}\|f''\|.
\end{aligned}\tag{23}$$

Combining Theorem 1, (14) and (23) we get

Theorem 5. *Define* $n_\alpha:=n+2(\alpha+1)$. *For* $t\geqslant0$, $n\geqslant N(\alpha)$, $k\geqslant1$, $f\in C^2[0,1]$,

$$\begin{aligned}
&\|M_{n,\alpha}^k f-T(t)f\|\\
&\quad\leqslant t\|n_\alpha(M_{n,\alpha}f-f)-A_\alpha f\|+\left(\left|t-\frac{k}{n_\alpha}\right|+\frac{\sqrt{k}}{n_\alpha}\right)\|n_\alpha(M_{n,\alpha}f-f)\|\\
&\quad\leqslant\frac{t}{4}\cdot\frac{n+\alpha+1}{n+2\alpha+3}\tilde{\omega}\left(f'';\sqrt{\frac{1}{6}\cdot\frac{n+\alpha+2}{(n+2\alpha+4)_2}}\right)\\
&\qquad+\left(\left|t-\frac{k}{n_\alpha}\right|+\frac{\sqrt{k}}{n_\alpha}\right)(\alpha+1)\|f'\|+\left[\left(\left|t-\frac{k}{n_\alpha}\right|+\frac{\sqrt{k}}{n_\alpha}\right)\frac{n_\alpha}{4}+C(\alpha)t\right]\\
&\qquad\times\frac{1}{n+2\alpha+3}\|f''\|.
\end{aligned}$$

Setting $k:=[n_\alpha t]$ we derive

Corollary 6. *For* $t\geqslant0$, $n\geqslant N(\alpha)$ *and* $f\in C^2[0,1]$,

$$\begin{aligned}
\|M_{n,\alpha}^{[n_\alpha t]}f-T(t)f\|\leqslant{}&\frac{t}{4}\tilde{\omega}\left(f'';\sqrt{\frac{1}{6n_\alpha}}\right)+\left(\frac{1}{n_\alpha}+\sqrt{\frac{t}{n_\alpha}}\right)(\alpha+1)\|f'\|\\
&+\frac{1}{4}\left(\frac{4C(\alpha)t+1}{n_\alpha}+\sqrt{\frac{t}{n_\alpha}}\right)\|f''\|.
\end{aligned}\tag{24}$$

5. Supplementary considerations for small values of n

In this section we consider the cases $1 \leqslant n < N(\alpha)$ for some selected values of α. Moreover, we state a conjecture which we confirm for $\alpha \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}\}$.

5.1. $\alpha = -1$ *(genuine Bernstein–Durrmeyer operators)*

In this case we have $N(\alpha) = 7$. By (17) we get for all n

$$\frac{M_{n,-1}(e_1 - xe_0)^4(x)}{M_{n,-1}(e_1 - xe_0)^2(x)} = 6\frac{(n-7)X + 2}{(n+2)_2}, \quad n \geqslant 1.$$

This function is convex for $1 \leqslant n < 7$, constant for $n = 7$ and concave for $n > 7$ from where we conclude that for $1 \leqslant n < 7 = N(-1)$

$$\frac{M_{n,-1}(e_1 - xe_0)^4(x)}{M_{n,-1}(e_1 - xe_0)^2(x)} \leqslant \frac{M_{n,-1}(e_1 - xe_0)^4(0)}{M_{n,-1}(e_1 - xe_0)^2(0)} = \frac{12}{(n+2)_2}. \tag{25}$$

For $\alpha > -1$ the situation is more complicated. For these cases we have the following.

Conjecture 7. *Let $\alpha > -1$. Then there is a natural number $n(\alpha)$ such that for $n(\alpha) \leqslant n < N(\alpha)$, $x \in [0, 1]$*

$$\frac{M_{n,\alpha}(e_1 - xe_0)^4(x)}{M_{n,\alpha}(e_1 - xe_0)^2(x)} \leqslant (\alpha + 4)\frac{n + \alpha + 2}{(n + 2\alpha + 4)_2}. \tag{26}$$

For $\alpha = -1$ the conjecture can be confirmed with $n(\alpha) = 3$.

Before we discuss other values of α we note that all of them have the following in common.

The denominator of the ratio of moments is for all $n \in \mathbb{N}$ greater than 0. That is, after multiplying both sides of (26) by the denominator and sorting by powers of X shows that (26) is equivalent with

$$4p_4X^2 \leqslant 2[(\alpha + 4)(n + \alpha + 2)p_1 - 2p_5]X + (\alpha + 4)(n + \alpha + 2)p_2 - p_6. \tag{27}$$

Furthermore, for the constant term on the right-hand side we have

$$(\alpha + 4)(n + \alpha + 2)p_2 - p_6 \geqslant 0 \quad \text{for all } n \in \mathbb{N}. \tag{28}$$

5.2. $\alpha = -\frac{1}{2}$ *(Chebyshev case of the first kind)*

This case corresponds—speaking in terms of orthogonal polynomials—to Chebyshev polynomials of the first kind. Here we have $n(\alpha) = 2$ and our conjecture reads as follows: For $2 \leqslant n \leqslant 12$ there holds

$$\frac{M_{n,-1/2}(e_1 - xe_0)^4(x)}{M_{n,-1/2}(e_1 - xe_0)^2(x)} \leqslant \frac{7}{4} \cdot \frac{2n+3}{(n+3)_2}$$

which is equivalent to

$$12(n^2 - 13n + 2)X^2 \leqslant \left(7n^2 - \frac{83}{2}n + \frac{39}{2}\right)X + \frac{21}{8}(n-1). \tag{29}$$

For $2 \leqslant n \leqslant 12$ it is true that $n^2 - 13n + 2 < 0$ and for $6 \leqslant n \leqslant 12$ one has $7n^2 - \frac{83}{2}n + \frac{39}{2} > 0$.

Thus the conjecture is true for $6 \leqslant n \leqslant 12$.

For each $n = 2, 3, 4, 5$ it is easy to verify that the quadratic polynomial defined as the difference of the right-hand side and the left-hand side of (29) has positive leading coefficient and has no real roots, which shows that (29) holds for these values of n as well.

5.3. $\alpha = 0$ *(Legendre case)*

Now $n(\alpha) = 1$, and the conjecture reads as follows: For $1 \leqslant n \leqslant 20$ one has

$$\frac{M_{n,0}(e_1 - xe_0)^4(x)}{M_{n,0}(e_1 - xe_0)^2(x)} \leqslant 4 \cdot \frac{n+2}{(n+4)_2}$$

which is equivalent with

$$3(n^2 - 21n + 10)X^2 \leqslant 2(n^2 - 10n + 9)X + 2(n-1). \tag{30}$$

For $1 \leqslant n \leqslant 20$ it is true that $n^2 - 21n + 10 < 0$, and for $n = 1$ and $9 \leqslant n \leqslant 20$ it is true that $n^2 - 10n + 9 > 0$.

This confirms the conjecture for $n = 1$ and $9 \leqslant n \leqslant 20$.

For $2 \leqslant n \leqslant 8$ a similar discussion as in case $\alpha = -\frac{1}{2}$, i.e., the consideration of the difference of the right-hand side and the left-hand side of (30) gives the desired result.

5.4. $\alpha = \frac{1}{2}$ *(Chebyshev case of the second kind)*

In this case again $n(\alpha) = 1$. The conjecture now reads: For $1 \leqslant n \leqslant 30$ one has

$$\frac{M_{n,1/2}(e_1 - xe_0)^4(x)}{M_{n,1/2}(e_1 - xe_0)^2(x)} \leqslant \frac{9}{4} \cdot \frac{2n+5}{(n+5)_2},$$

being equivalent with

$$12(n^2 - 31n + 30)X^2 \leqslant \left(9n^2 - \frac{273}{2}n + 180\right)X + \frac{135}{8}(n-1). \tag{31}$$

For $1 \leqslant n \leqslant 30$ we get $n^2 - 31n + 30 \leqslant 0$ and for $n = 1$ and $14 \leqslant n \leqslant 30$ there holds: $9n^2 - \frac{273}{2}n + 180 > 0$.

Thus the claim is true for $n = 1$ and $14 \leqslant n \leqslant 30$.

For $2 \leqslant n \leqslant 13$ the proof is again similar to the analogous cases in case $\alpha = -\frac{1}{2}$.

6. Estimates for continuous functions

Here we show how the C^2 estimates can be carried over to $C[0,1]$. To this end we use

Lemma 8 (*see Gonska [15]*). *Let $I = [0,1]$ and $f \in C^r(I)$, $r \in \mathbb{N}_0$. For any $h \in (0,1]$ and $s \in \mathbb{N}$ there exists a function $f_{h,r+s} \in C^{2r+s}(I)$ with*

(i) $\|f^{(j)} - f^{(j)}_{h,r+s}\| \leqslant c\omega_{r+s}(f^{(j)}; h)$ *for* $0 \leqslant j \leqslant r$,

(ii) $\|f_{h,r+s}^{(j)}\| \leqslant ch^{-j}\omega_j(f;h)$, *for* $0 \leqslant j \leqslant r+s$,
(iii) $\|f_{h,r+s}^{(j)}\| \leqslant ch^{-(r+s)}\omega_{r+s}(f^{(j-r-s)};h)$, *for* $r+s \leqslant j \leqslant 2r+s$.
Here the constant $c = c_{r,s}$ *depends only on* r *and* s.

We will use the lemma for $r = 0$, $s = 3$ to obtain functions $f_{h,3}$ satisfying

(i) $\|f - f_{h,3}\| \leqslant c\omega_3(f;h)$;
(ii) $\|f'_{h,3}\| \leqslant ch^{-1}\omega_1(f;h)$;
(iii) $\|f''_{h,3}\| \leqslant ch^{-2}\omega_2(f;h)$;
(iv) $\|f'''_{h,3}\| \leqslant ch^{-3}\omega_3(f;h)$.

Theorem 9. *Let* $M_{n,\alpha}^{[n_\alpha t]}$, $T(t)$ *be given as above*, $f \in C[0,1]$, $n \geqslant N(\alpha)$. *Then*

$$\begin{aligned}
&\|M_{n,\alpha}^{[n_\alpha t]} f - T(t)f\| \\
&\quad \leqslant c\left\{\left(2 + \frac{t}{4\sqrt{6}}\right)\omega_3(f;n_\alpha^{-1/6}) + \frac{1}{4}\left(\frac{4C(\alpha)t+1}{\sqrt{n_\alpha}} + \sqrt{t}\right)n_\alpha^{-1/6}\omega_2(f;n_\alpha^{-1/6})\right. \\
&\qquad \left. + \left(\frac{1}{\sqrt{n_\alpha}} + \sqrt{t}\right)(\alpha+1)n_\alpha^{-1/3}\omega_1(f;n_\alpha^{-1/6})\right\}.
\end{aligned}$$

Proof. For $f \in C[0,1]$ and $g \in C^3[0,1]$ decompose as follows:

$$\|M_{n,\alpha}^{[n_\alpha t]} f - T(t)f\| \leqslant \|M_{n,\alpha}^{[n_\alpha t]}(f-g) - T(t)(f-g)\| + \|M_{n,\alpha}^{[n_\alpha t]} g - T(t)g\|.$$

The second summand was considered earlier in Corollary 6. Moreover, using the fact that $M_{n,\alpha}^{[n_\alpha t]}$ and $T(t)$ are contractions, we arrive at the upper bound ($n \geqslant N(\alpha)$)

$$\begin{aligned}
&\|M_{n,\alpha}^{[n_\alpha t]} f - T(t)f\| \\
&\quad \leqslant 2\|f-g\| + \frac{t}{4}\tilde{\omega}\left(g''; \frac{1}{\sqrt{6n_\alpha}}\right) + \frac{1}{4}\left(\frac{4C(\alpha)t+1}{n_\alpha} + \sqrt{\frac{t}{n_\alpha}}\right)\|g''\| \\
&\qquad + \left(\frac{1}{n_\alpha} + \sqrt{\frac{t}{n_\alpha}}\right)(\alpha+1)\|g'\|.
\end{aligned}$$

For $g \in C^3[0,1]$ one has (see Remark 3(ii)) $\tilde{\omega}(g''; \frac{1}{\sqrt{6n_\alpha}}) \leqslant \frac{1}{\sqrt{6n_\alpha}}\|g'''\|$.

Moreover, taking $g = f_{h,3}$ from above gives, $c = c_{r,s}$ being the constant from Lemma 8,

$$\begin{aligned}
&\|M_{n,\alpha}^{[n_\alpha t]} f - T(t)f\| \\
&\quad \leqslant c\left\{\left(2 + \frac{t}{4}\frac{1}{\sqrt{6n_\alpha}}h^{-3}\right)\omega_3(f;h) + \frac{1}{4}\left(\frac{4C(\alpha)t+1}{n_\alpha} + \sqrt{\frac{t}{n_\alpha}}\right)h^{-2}\omega_2(f;h)\right. \\
&\qquad \left. + \left(\frac{1}{n_\alpha} + \sqrt{\frac{t}{n_\alpha}}\right)(\alpha+1)h^{-1}\omega_1(f;h)\right\}.
\end{aligned}$$

For $h = n_\alpha^{-1/6}$ we arrive at the desired estimate. □

We now discuss the cases $\alpha = -1$ and 0 further.

(i) $t \geqslant 0$, $\alpha = -1$, so that $C(-1) = \frac{1}{4}$ and $n_\alpha = n$. In this case we obtain

$$\begin{aligned} &\|M_{n,-1}^{[nt]} f - T(t)f\| \\ &\quad\leqslant c\left\{\left(2+\frac{t}{4\sqrt{6}}\right)\omega_3(f;n^{-1/6}) + \frac{1}{4}\left(\frac{t+1}{\sqrt{n}}+\sqrt{t}\right)n^{-1/6}\omega_2(f;n^{-1/6})\right\}. \end{aligned} \tag{32}$$

Note that the term with the first order modulus disappears, for which the real reason is that $M_{n,-1}$ reproduces linear functions.

Only for this case we discuss what estimate (32) from above implies for smooth functions. Let $f \in C^3[0,1]$. Then

$$\begin{aligned} \|M_{n,-1}^{[nt]} f - T(t)f\| &\leqslant c\left\{\left(2+\frac{t}{4\sqrt{6}}\right)n^{-1/2}\|f'''\| + \frac{1}{4}\left(\frac{t+1}{\sqrt{n}}+\sqrt{t}\right)n^{-1/2}\|f''\|\right\} \\ &= \mathcal{O}(n^{-1/2}). \end{aligned}$$

Note that this order was also the one we started off with, that is, before we carried the results over to $C[0,1]$ (see the proof of Theorem 9).

(ii) $t \geqslant 0$, $\alpha = 0$. Now $C(0) = 1$, $n_\alpha = n+2$, and the inequality in this case reads

$$\begin{aligned} &\|M_{n,0}^{[(n+2)t]} f - T(t)f\| \\ &\quad\leqslant c\left\{\left(2+\frac{t}{4\sqrt{6}}\right)\omega_3(f;(n+2)^{-1/6})\right. \\ &\qquad + \frac{1}{4}\left(\frac{4t+1}{\sqrt{n+2}}+\sqrt{t}\right)(n+2)^{-1/6}\omega_2(f;(n+2)^{-1/6}) \\ &\qquad \left. + \left(\frac{1}{\sqrt{n+2}}+\sqrt{t}\right)(n+2)^{-1/3}\omega_1(f;(n+2)^{-1/6})\right\} \\ &\quad\leqslant c(2+\max\{\sqrt{t},t\})\sum_{k=0}^{2}(n+2)^{-k/6}\omega_{3-k}(f;(n+2)^{-1/6}). \end{aligned}$$

(iii) For $t \geqslant 0$ and $\alpha = -\frac{1}{2}$ or $\frac{1}{2}$ we obtain upper bounds which are essentially the same as the one for $\alpha = 0$.

Acknowledgment

The authors express their gratitude to the referees for their helpful constructive remarks.

References

[1] U. Abel, M. Heilmann, The complete asymptotic expansion for Bernstein–Durrmeyer operators with Jacobi weights, Mediterr. J. Math. 1 (4) (2004) 487–499.

[2] F. Altomare, M. Campiti, Korovkin-type Approximation Theory and its Applications, Walter de Gruyter, Berlin, 1994.

[3] F. Altomare, I. Raşa, On a class of exponential-type operators and their limit semigroups, J. Approx. Theory 135 (2) (2005) 258–275.

[4] F. Altomare, I. Raşa, On some classes of diffusion equations and related approximation problems, in: D. Mache, J. Szabados, M.G. de Bruin (Eds.), Trends and Applications in Constructive Approximation, Proceedings of the 4th IBoMAT Conference, Witten-Bommerholz, 2004, Birkhäuser, Basel, 2005, pp. 13–26.

[5] H. Berens, Y. Xu, On Bernstein–Durrmeyer polynomials with Jacobi-weights, in: Approximation Theory and Functional Analysis, College Station, TX, 1990, Academic Press, Boston, MA, 1991, pp. 25–46.

[6] H. Berens, Y. Xu, On Bernstein–Durrmeyer polynomials with Jacobi-weights: the cases $p = 1$ and $p = \infty$. Approximation, interpolation and summability, Ramat Aviv, 1990/Ramat Gan, 1990, Israel Mathematical Conference Proceedings, vol. 4, Bar-Ilan University, Ramat Gan, 1991, pp. 51–62.
[7] P. Butzer, H. Berens, Semi-groups of Operators and Approximation, Springer, New York, 1967.
[8] P.L. Butzer, W. Dickmeis, Hu. Jansen, R.J. Nessel, Alternative forms with orders of the Lax equivalence theorem in Banach spaces, Computing 17 (4) (1977) 335–342.
[9] P.L. Butzer, R. Weis, On the Lax equivalence theorem equipped with orders, J. Approx. Theory 19 (3) (1977) 239–252.
[10] M. Campiti, Cr. Tacelli, Rate of convergence in the Trotter's approximation theorem, preprint, 2007.
[11] M. Campiti, Cr. Tacelli, Quantitative approximation of semigroups and resolvent operators by iterates of Stancu operators, in: M. Neamtu, et al. (Eds.), Approximation Theory XII, Proceedings of the 12th International Conference, San Antonio, TX, USA, March 4–8, 2007, Modern Methods in Mathematics, Nashboro Press, Brentwood, TN, 2008, pp. 50–59.
[12] M.M. Derriennic, Sur l'approximation de fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifiés, J. Approx. Theory 31 (4) (1981) 325–343.
[13] W. Dickmeis, R.J. Nessel, Classical approximation processes in connection with Lax equivalence theorems with orders, Acta Sci. Math. (Szeged) 40 (1–2) (1978) 33–48.
[14] K.J. Engel, R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Springer, New York, 2000.
[15] H. Gonska, Degree of approximation by lacunary interpolators: $(0, \ldots, R-2, R)$ interpolation, Rocky Mountain J. Math. 19 (1) (1989) 157–171.
[16] H. Gonska, On the degree of approximation in Voronovskaja's theorem, Studia Univ. Babes-Bolyai Math. 52 (3) (2007) 103–116.
[17] H. Gonska, D. Kacsó, I. Raşa, On genuine Bernstein–Durrmeyer operators, Result. Math. 50 (2007) 213–225.
[18] H. Gonska, P. Piţul, I. Raşa, On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators, in: O. Agratini, P. Blaga (Eds.), Numerical Analysis and Approximation Theory, Proceedings of the International Conference on NAAT, Cluj-Napoca, Casa Cărţii de ştiinţă, Cluj-Napoca, 2006, pp. 55–80.
[19] H. Gonska, I. Raşa, The limiting semigroup of the Bernstein operators: degree of convergence, Acta Math. Hungary 111 (1–2) (2006) 119–130.
[20] D. Kacsó, Certain Bernstein–Durrmeyer type operators preserving linear functions, Habilitationsschrift, University of Duisburg–Essen, 2006.
[21] S. Karlin, Z. Ziegler, Iteration of positive approximation operators, J. Approx. Theory 3 (3) (1970) 310–339.
[22] A. Lupaş, The approximation by means of some linear positive operators, in: M.W. Müller, M. Felten, D. Mache (Eds.), Approximation Theory, Proceedings of the International Dortmund Meeting IDoMAT 95, Witten, Mathematical Research, vol. 86, Akademie-Verlag, Berlin, 1995, pp. 201–229.
[23] E. Mangino, I. Raşa, A quantitative version of Trotter's approximation theorem, J. Approx. Theory 146 (2007) 149–156.
[24] R. Păltănea, Sur un opérateur polynomial defini sur l'ensemble des fonctions intégrables, "Babes-Bolyai" Univ. Fac. Math. Res. Semin. 2 (1983) 101–106.
[25] P.E. Parvanov, B.D. Popov, The limit case of Bernstein's operators with Jacobi-weights, Math. Balkanica (N.S.) 8 (2–3) (1994) 165–177.
[26] I. Raşa, Semigroups associated to Mache operators, in: M. Buhmann, D. Mache (Eds.), Advanced Problems in Constructive Approximation, Proceedings of the 3rd International Dortmund Meeting IDoMAT 2001, Witten, 2001, Birkhäuser, Basel, 2003, pp. 143–152.
[27] Th. Sauer, Ein Bernstein–Durrmeyer-Operator auf dem Simplex, Dissertation Universität Erlangen-Nürnberg, 1993.
[28] G. Tachev, Voronovskaja's theorem revisited, J. Math. Anal. Appl. 343 (1) (2008) 399–404.
[29] T. Vladislav, I. Raşa, Analiză Numerică: Aproximare, problema lui Cauchy abstractă, proiectori Altomare, Editura Technică, Bucureşti, 1999.